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Interfacial capillary–gravity waves due to a fundamental singularity in a system of two semi-infinite fluids

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Abstract The interfacial capillary–gravity waves due to a transient fundamental singularity immersed in a system of two semi-infinite immiscible fluids of different densities are investigated analytically for two- and three-dimensional cases. The two-fluid system, which consists of an inviscid fluid overlying a viscous fluid, is assumed to be incompressible and initially quiescent. The two fluids are each homogeneous, and separated by a sharp and stable interface. The Laplace equation is taken as the governing equation for the inviscid flow, while the linearized unsteady Navier–Stokes equations are used for the viscous flow. With surface tension taken into consideration, the kinematic and dynamic conditions on the interface are linearized for small-amplitude waves. The singularity is modeled as a simple mass source when immersed in the inviscid fluid above the interface, or as a vertical point force when immersed in the viscous fluid beneath the interface. Based on the integral solutions for the interfacial waves, the asymptotic wave profiles are derived for large times with a fixed distance-to-time ratio by means of the generalized method of stationary phase. It is found that there exists a minimum group velocity, and the wave system observed will depend on the moving speed of the observer. Two schemes of expansion of the phase function are proposed for the two cases when the moving speed of an observer is larger than, or close to the minimum group velocity. Explicit analytical solutions are presented for the long gravity-dominant and the short capillary-dominant wave systems, incorporating the effects of density ratio, surface tension, viscosity and immersion depth of the singularity.

Keywords Asymptotic solution · Fundamental singularity · Interfacial wave · Surface tension · Viscosity

1 Introduction

The singularity method is a powerful analytical approach to study viscous flows induced by fluid–body interaction when the inertial effect is small [1–3]. The Stokes or Oseen equations have long been employed to simulate such flows. The analytical solutions for the fundamental singularities of the Stokes and Oseen flows in unbounded

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domains, or the so-called Stokeslet and Oseenlet, have been extensively examined [1,4–7]. Recently, Venkatalaxmi et al. [6] showed that the solution for an unsteady Stokeslet can be expressed in a general form that involves two scalar functions. Shu and Chwang [7] derived analytical solutions for the generalized unsteady Stokeslet and Oseenlet.

The aforementioned works were conducted for an unbounded viscous fluid. Several analytical attempts have been made for the study of singularity-induced waves, yielding simple but useful models for the laminar interaction of viscous wakes with a free surface due to a submerged body maneuvering in a real fluid. The characteristics of the unsteady free-surface waves due to a Stokeslet and an Oseenlet in a single fluid of infinite depth were discussed in detail by Lu and Chwang in [8] and [9], respectively (see also the references therein). It is found that the non-physical behaviors arising from two classical problems in the theory of water waves, namely the Cauchy–Poisson waves (CPW) in a stationary fluid and the Neumann–Kelvin waves (NKW) in a uniformly running stream, as described by Chen and Wu [10], can be remedied by the introduction of the Stokeslet and Oseenlet [8,9]. Viscosity plays a vital role in the modeling of singularity-induced waves in a real fluid. Firstly, the asymptotic locations of the zeros of the dispersion equation in the upper or lower half-plane ensure that the NKW appear in the downstream or upstream region. Secondly, the singular behavior, as predicted by the potential theory, of infinite amplitudes for the CPW in the near region, and for the diverging component of the NKW near the moving path of the pressure point, can be removed thanks to the presence of viscous decay factors. Thirdly, in contrast to the non-vanishing inviscid transient CPW and NKW, the viscous transient waves will die out as time goes to infinity owing to the presence of temporal decay factors. Thus, an ultimate steady state can be attained, which is consistent with the physical reality. Based on the linearized potential theory for an inviscid fluid, Raphaël and de Gennes [11] found that the wave resistance experienced by a disturbance moving at the free surface of a still fluid becomes unbounded as the velocity of disturbance V decreases towards the minimum phase speed $c^{\min}$ of the capillary–gravity waves and the wave resistance is zero when $V < c^{\min}$. However, two important features can be revealed by the use of the linearized Navier–Stokes equations for a viscous fluid [12]: (a) the wave resistance increases steeply near $V = c^{\min}$ and remains bounded; (b) the wave resistance takes finite values when $0 < V < c^{\min}$. These features are consistent with experimental results [12].

Recently, the steady viscous interfacial NKW and the unsteady inviscid interfacial CPW in a system of two semi-infinite immiscible fluids of different densities were investigated analytically by Lu and Chwang [13,14] and Lu et al. [15], respectively. The effect of density ratio on the wave patterns was discussed in detail. The motion of a particle near the interface between two fluids is of fundamental interest because of its importance in many engineering applications [16]. In the present study, the unsteady viscous interfacial CPW due to a fundamental singularity in a system of two semi-infinite fluids are considered. The system, which consists of an upper inviscid fluid containing an instantaneous simple source and a lower viscous fluid containing a vertical point force, is assumed to be incompressible, homogeneous and initially quiescent. The fluid inertia is assumed to be negligible when the fluids are set into motion by the small perturbations. The Laplace equation is used as the governing equation for the inviscid fluid motion above the interface. On the other side, the unsteady Stokes equations are employed for the viscous fluid motion below the interface. Accordingly, the point force is mathematically represented by an unsteady Stokeslet. As remarked by Prosperetti [17], the use of the unsteady Stokes equations is justified as long as the wave amplitudes and the fluid velocity are small compared with the typical wavelength and the wave velocity, respectively. Based on the unsteady Stokes equations, Miles [18] and Debnath [19] considered the concentric waves due to a free-surface impulse in a viscous fluid, Prosperetti [20] studied the effect of viscosity on the two-dimensional sinusoidal standing waves, and Wu et al. [21] looked into the viscous effect on the transient free surface flow in a two-dimensional tank.

The non-physical behavior of infinite amplitudes for the CPW and NKW, as predicted by the potential theory, can be eliminated upon taking into account surface tension in the boundary conditions, as has been demonstrated in the numerical analyses by Chen and Duan [22] and Chen [23]. A preliminary study on the combined effects of viscosity and surface tension on the waves due to a Stokeslet applied at the free-surface and due to a simple source has been performed by Chen et al. [24,25]. In this paper, the interfacial wave motion for large times with a fixed distance-to-time ratio will be analyzed with the aid of Laplace–Fourier integral transforms and a generalized method of stationary phase. Based on the assumption that the wave amplitude is very small compared with the wavelength, the kinematic and dynamic boundary conditions are linearized with surface tension taken into consideration. The general mathematical model for the two- and three-dimensional problems is formulated in Sect. 2. The analytical solutions for free-surface waves due to an instantaneous singularity are then obtained in Sect. 3 for the two-dimensional problem, and in Sect. 4 for the three-dimensional problem. Finally, discussion and conclusions are made in Sect. 5. One could expect that the interfacial displacements consist of the long gravity-dominant and short capillary-dominant waves. Based on the explicitly analytical solutions for the corresponding wave-numbers obtained in this paper, the principal physical features can analytically be investigated.

2 General mathematical formulation

The fundamental singularity for inviscid flow is a simple source, which represents a point-mass source in the fluid. Cartesian coordinates (x, y, z) are taken in such a way that the x- and y-axes are fixed on the undisturbed interface while the z-axis points vertically upwards. For the upper inviscid fluid, the governing equation is

$$\nabla^2 \Phi_{1n} = \Delta_{1n} M \delta(\mathbf{x} - \mathbf{x}_0)\delta(t), \quad (n = 1, 2), \tag{1}$$

where the first subscript stands for the fluid (1: upper; 2: lower) containing an observation point $\mathbf{x} = (x, y, z)$, while the second subscript stands for the fluid containing a source point $\mathbf{x}_0 = (0, 0, z_0)$. Φ_{1n} is the velocity potential for the perturbed flow in the upper fluid due to the singularity immersed in fluid n, Δ_{mn} the Kronecker delta, $\delta(\ )$ the Dirac delta function, and M the magnitude of the instantaneous simple source. It should be noted that for $n = 1$, $z_0 > 0$ while for $n = 2$, $z_0 < 0$.

The fundamental singularity in the linearized viscous flow is a Stokeslet, which represents a point force acting on the fluid. For the lower fluid, the governing equations are

$$\nabla \cdot \mathbf{u}_{2n} = 0, \tag{2}$$

$$\rho_2 \frac{\partial \mathbf{u}_{2n}}{\partial t} = -\nabla P_{2n} + \mu \nabla^2 \mathbf{u}_{2n} + \Delta_{2n} \mathbf{F} \delta(\mathbf{x} - \mathbf{x}_0)\delta(t), \tag{3}$$

where $\mathbf{u}_{2n} = (u_{2n}, v_{2n}, w_{2n})$ is the disturbed velocity field in the lower fluid due to a singularity immersed in fluid n, and P_{2n} the corresponding hydrodynamic pressure. ρ_2 and μ are the density and dynamic viscosity of the lower fluid, respectively. $\mathbf{F} = (0, 0, -F)$ is the point force acting on the lower fluid.

For small-amplitude waves, we may impose the linearized boundary conditions on the undisturbed interface $(z = 0)$. The kinematic conditions on the interface are given by

$$\frac{\partial \eta_n}{\partial t} = \frac{\partial \Phi_{1n}}{\partial z} = w_{2n}, \tag{4}$$

where η_n is the interface profile due to the singularity immersed in fluid n. The dynamic conditions on the interface are given by

$$\mu \left(\frac{\partial u_{2n}}{\partial z} + \frac{\partial w_{2n}}{\partial x} \right) = 0, \tag{5}$$

$$\mu \left(\frac{\partial v_{2n}}{\partial z} + \frac{\partial w_{2n}}{\partial y} \right) = 0, \tag{6}$$

$$-\rho_1 \left(\frac{\partial \Phi_{1n}}{\partial t} + g\eta_n \right) = P_{2n} - \rho_2 g \eta_n - 2\mu \frac{\partial w_{2n}}{\partial z} + T \left(\frac{\partial^2 \eta_n}{\partial x^2} + \frac{\partial^2 \eta_n}{\partial y^2} \right), \tag{7}$$

where ρ_1 is the density of the upper fluid, T the coefficient of the surface tension acting along the interface. Equation 5 represents the vanishing of shearing stress in the x-direction while Eq. 6 in the y-direction. Equation 7 represents the balance of the normal stresses. In addition, the initial values of the velocity, the hydrodynamic pressure and the interfacial elevation are taken to be

$$\Phi_{1n}|_{t=0} = P_{2n}|_{t=0} = \eta_n|_{t=0} = 0, \quad \mathbf{u}_{2n}|_{t=0} = \mathbf{0}. \tag{8}$$

The condition of linearity allows us to assume the perturbed flow being the sum of a singular and a regular flow, representing the effects of an instantaneous singularity, and of the interface, respectively. For the upper inviscid fluid,

$$\Phi_{1n} = \Delta_{1n}\Phi_{11}^S + \Phi_{1n}^R, \tag{9}$$

where Φ_{11}^S is the potential due to the singularity in an unbounded domain, while Φ_{1n}^R is a harmonic function everywhere in the bounded domain, and $\nabla^2\Phi_{1n}^R = 0$. For the lower viscous fluid, we write

$$[\mathbf{u}_{2n}, P_{2n}] = \Delta_{2n}[\mathbf{u}_{22}^S(\mathbf{x};\mathbf{x}_0), P_{22}^S(\mathbf{x};\mathbf{x}_0)] + [\mathbf{u}_{2n}^R(\mathbf{x}), P_{2n}^R(\mathbf{x})]. \tag{10}$$

Furthermore, the continuous vector $\mathbf{u}_{2n}^R$ is taken as the sum of an irrotational and a solenoidal vector, $\mathbf{u}_{2n}^R = \nabla\Phi_{2n}^R + \mathbf{V}_{2n}^T$, where Φ_{2n}^R is a harmonic function and $\mathbf{V}_{2n}^T$ a solenoidal vector. Thus,

$$\nabla^2\Phi_{2n}^R = 0, \tag{11}$$

$$\nabla\cdot\mathbf{V}_{2n}^T = 0, \tag{12}$$

$$\frac{\partial\mathbf{V}_{2n}^T}{\partial t} = \nu\nabla^2\mathbf{V}_{2n}^T, \tag{13}$$

$$P_{2n}^R = -\rho_2\frac{\partial\Phi_{2n}^R}{\partial t} + \phi(t), \tag{14}$$

where $\nu = \mu/\rho_2$, and $\phi(t)$ a function of t. Therefore, the boundary conditions can be expressed in terms of Φ_{1n}^R, Φ_{11}^S, $\mathbf{V}_{2n}^T$, Φ_{2n}^R, and $\mathbf{u}_{22}^S$ on the undisturbed interface ($z = 0$) as follows:

$$\frac{\partial\eta_n}{\partial t} - \frac{\partial\Phi_{1n}^R}{\partial z} = \Delta_{1n}\frac{\partial\Phi_{11}^S}{\partial z}, \tag{15}$$

$$\left(\frac{\partial\Phi_{2n}^R}{\partial z} + w_{2n}^T\right) - \frac{\partial\Phi_{1n}^R}{\partial z} = \Delta_{1n}\frac{\partial\Phi_{11}^S}{\partial z} - \Delta_{2n}w_{22}^S, \tag{16}$$

$$2\frac{\partial^2\Phi_{2n}^R}{\partial x\partial z} + \frac{\partial u_{2n}^T}{\partial z} + \frac{\partial w_{2n}^T}{\partial x} = -\Delta_{2n}\left(\frac{\partial u_{22}^S}{\partial z} + \frac{\partial w_{22}^S}{\partial x}\right), \tag{17}$$

$$2\frac{\partial^2\Phi_{2n}^R}{\partial y\partial z} + \frac{\partial v_{2n}^T}{\partial z} + \frac{\partial w_{2n}^T}{\partial y} = -\Delta_{2n}\left(\frac{\partial v_{22}^S}{\partial z} + \frac{\partial w_{22}^S}{\partial y}\right), \tag{18}$$

$$\begin{aligned}(1-\sigma)g\eta_n - \sigma\frac{\partial\Phi_{1n}^R}{\partial t} + \frac{\partial\Phi_{2n}^R}{\partial t} - \frac{\phi(t)}{\rho_2} + 2\nu\left(\frac{\partial^2\Phi_{2n}^R}{\partial z^2} + \frac{\partial w_{2n}^T}{\partial z}\right) - \tau\left(\frac{\partial^2\eta_n}{\partial x^2} + \frac{\partial^2\eta_n}{\partial y^2}\right)\\ = \Delta_{1n}\sigma\frac{\partial\Phi_{11}^S}{\partial t} + \Delta_{2n}\left(\frac{P_{22}^S}{\rho_2} - 2\nu\frac{\partial w_{22}^S}{\partial z}\right),\end{aligned} \tag{19}$$

where $\tau = T/\rho_2$, $\sigma = \rho_1/\rho_2$. It should be noted that the physical units for M and F in two and three dimensions are different. The mathematical formulation Eqs. 1–19 will reduce to the two-dimensional case when $v = 0$ and $\partial/\partial y = 0$.

3 Two-dimensional interfacial waves due to an instantaneous singularity

For two-dimensional cases, the fundamental solution to the Laplace equation can be given by

$$\Phi_{11}^{S}(\mathbf{x};\mathbf{x_0}) = -\frac{M\delta(t)}{2\pi}\log\frac{1}{r} = \frac{M}{8\pi^2 \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}\alpha\frac{1}{k}\exp(-k|z-z_0|+\mathrm{i}\alpha x+st), \tag{20}$$

where $r=||\mathbf{x}-\mathbf{x}_0||$, and $k=|\alpha|$. The fundamental solution to the unsteady Stokes equations can be written as

$$\mathbf{u}_{22}^{S}(\mathbf{x};\mathbf{x_0}) = \frac{1}{8\pi^2\rho_2 \mathrm{i}}\mathbf{F}\cdot(\nabla\nabla-\mathbf{I}\nabla^2)\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}\alpha\frac{1}{s}\exp(\mathrm{i}\alpha x)\left[\frac{1}{k}\exp(-k|z-z_0|)\right.$$
$$\left.-\frac{1}{b}\exp(-b|z-z_0|)\right], \tag{21}$$

$$P_{22}^{S}(\mathbf{x};\mathbf{x_0}) = -\frac{1}{8\pi^2\mathrm{i}}(\mathbf{F}\cdot\nabla)\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\mathrm{d}\alpha\frac{1}{k}\exp(\mathrm{i}\alpha x-k|z-z_0|), \tag{22}$$

where $\mathbf{I}$ is a unit tensor of rank two, and $b=\sqrt{s/\nu+k^2}$.

In order to obtain a formal solution for the interfacial wave motion, a joint Fourier–Laplace integral transform is introduced as follows:

$$[\tilde{\eta},\tilde{\Phi}_{1n}^{R},\tilde{\Phi}_{2n}^{R},\tilde{\mathbf{V}}_{2n}^{T}] = \int_{0}^{+\infty}\mathrm{d}t\int_{-\infty}^{+\infty}\mathrm{d}\alpha\exp(-\mathrm{i}\alpha x-st)\left[\eta,\Phi_{1n}^{R}\exp(kz),\Phi_{2n}^{R}\exp(-kz),\mathbf{V}_{2n}^{T}\exp(-bz)\right] \tag{23}$$

which is in accordance with Eqs. 11 and 13. Applying this transform to the left-hand sides of Eqs. 12 and 15–19, while substituting Eqs. 20–22 in the right-hand sides of these equations, we obtain a system of linear equations for the unknown functions $[\tilde{\eta},\tilde{\Phi}_{1n}^{R},\tilde{\Phi}_{2n}^{R},\tilde{\mathbf{V}}_{2n}^{T}]$. The function $\phi(t)=-\rho_2\Phi_{2n}^{R}(x,y,0,0)\delta(t)$ is imposed in order to satisfy the initial conditions (8). Upon some mathematical manipulation, the integral expression for the two-dimensional wave profile can be written as

$$\eta_n = \frac{1}{4\pi^2\mathrm{i}}\sum_{l=1}^{2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{0}^{+\infty}\mathrm{d}k\frac{A_n}{sD}\exp[(-1)^{l+1}\mathrm{i}kx+st], \tag{24}$$

where

$$A_1 = -M\sigma s^2\exp(-kz_0), \tag{25}$$

$$A_2 = -\frac{F}{\rho_2}[k(s+2\nu k^2)\exp(kz_0)-2\nu k^3\exp(bz_0)], \tag{26}$$

$$D = (1-\sigma)gk+\tau k^3+(1+\sigma)s^2+4\nu k^2 s+4\nu^2k^3(k-b). \tag{27}$$

Here D can be regarded as the dispersion function for the interfacial viscous capillary–gravity waves generated by a fundamental singularity in the system considered here.

The integral expression (24) represents the solution for the wave elevation due to the impulsive motion of a submerged singularity. The physical characteristics of the wave motion, however, are not apparent in these integral representations. Moreover, the exact evaluation of the integrals valid for all times is extremely difficult, and in general can only be performed numerically. In order to reveal the principal physical features of the wave motion, it is useful and necessary to perform an asymptotic analysis for the wave integral. In the following sections, the asymptotic behaviors of Eq. 24 will be studied for large t with x/t held fixed. This kind of asymptotic analysis has been applied to the classical Cauchy–Poisson problems [18,19,26,27], and to the wave system in an inviscid fluid with an ice-cover [28].

The first step is to use the Cauchy residue theorem to invert the Laplace transform in Eq. 24. According to Miles [18], the plane can be cut along $s=(-\infty,-\nu k^2]$. Thus, D has two zeros with respect to s in the cut plane, which can be determined by the dispersion equation $D = 0$. For small ν, the asymptotic solutions of $D = 0$ take the form of

$$s_j = (-1)^{j+1}\mathrm{i}\omega - 2\lambda\nu k^2 + o(\nu k^2), \quad (j = 1, 2), \tag{28}$$

where

$$\omega(k) = (\gamma g k + \lambda\tau k^3)^{1/2}, \tag{29}$$

$$\gamma = (1-\sigma)/(1+\sigma), \tag{30}$$

$$\lambda = 1/(1+\sigma). \tag{31}$$

The parameter γ is known as the Atwood number. By taking a contour integration in the complex s-plane, we can represent Eq. 24 by

$$\eta_n \approx -\frac{\lambda}{4\pi}\sum_{l=1}^{2}\sum_{j=1}^{2}\int_0^{+\infty}\frac{A_{nj}}{\omega^2}\exp\left(-2\nu k^2 t + \mathrm{i}t\Theta_{lj}\right)\mathrm{d}k, \tag{32}$$

where

$$A_{nj} = A_n|_{s=s_j}, \tag{33}$$

$$\Theta_{lj} = (-1)^{l+1}kx/t + (-1)^{j+1}\omega. \tag{34}$$

For the k integration in Eq. 32, the method of stationary phase is used for large t with x/t held fixed. The dominant contribution to the integral in Eq. 32 stems from the stationary points of the oscillatory factors of the integrand. It is easily seen that for $x > 0$, Θ_{12} and Θ_{21} have a stationary point while for $x < 0$, Θ_{11} and Θ_{22} have a stationary point. The stationary points for both $x > 0$ and $x < 0$ are the same, which can be determined by

$$\frac{\partial\Theta}{\partial k} = \frac{|x|}{t} - C_g = \frac{|x|}{t} - \frac{1}{2}(G + 3k^2)\left(\frac{\lambda\tau}{Gk + k^3}\right)^{1/2} = 0, \tag{35}$$

where $\Theta = k|x|/t - \omega$, $C_g(k) = \partial\omega/\partial k$, and $G = \gamma g/\lambda\tau$. Figure 1 shows curves for the group velocities $C_g(k)$. For simplicity, the ISO units are used for all the physical variables in this paper. It can be seen from Fig. 1 that there exists a minimum group velocity, denoted by $C_{g\min}(\sigma, \tau) = C_g(k_c)$, at which Eq. 35 has one real positive root $k_c(\sigma, \tau)$ only. It is clear that k_c is determined by

$$\omega_c'' = \frac{\partial^2\omega(k_c)}{\partial k^2} = 0. \tag{36}$$

The real positive root of Eq. (36) is exactly given by

$$k_c(\sigma, \tau) = \sqrt{\left(\frac{2}{\sqrt{3}} - 1\right)G}. \tag{37}$$

Thus, we have

$$C_{g\min}(\sigma, \tau) = C_g(k_c) = \omega_c' = (\sqrt{3} - 1)\sqrt{\frac{3\lambda\tau}{2}}\left(\frac{G}{2\sqrt{3} - 3}\right)^{1/4}. \tag{38}$$

When $|x|/t > C_{g\min}$, Eq. 35 has two real positive roots, $k_1(|x|/t, \sigma, \tau)$ and $k_2(|x|/t, \sigma, \tau)$ with $0 < k_1 < k_2 < +\infty$. Approximate solutions for $k_1(|x|/t, 0, \tau)$ and $k_2(|x|/t, 0, \tau)$ have been provided by Chen and Duan [22, Eqs. 6 and 8] for the free-surface capillary–gravity waves in an inviscid fluid. With the aid of a computer-algebra software package for symbolic calculation, for example, Mathematica®, the exact solutions for $k_1(|x|/t, \sigma, \tau)$ and $k_2(|x|/t, \sigma, \tau)$ can readily be obtained as

$$k_m = \frac{1}{36}[X + \sqrt{Q_4} + (-1)^m\sqrt{2Q_5}], \quad (m = 1, 2), \tag{39}$$

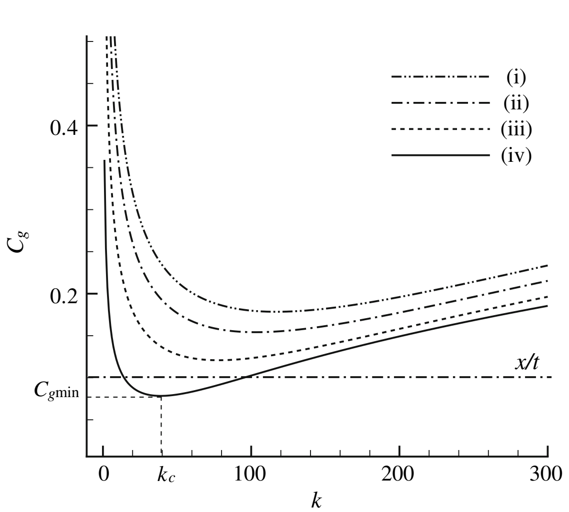

Fig. 1 Group velocity curves $C_g(k)$ with (i) $\sigma = 0.1$, (ii) $\sigma = 0.3$, (iii) $\sigma = 0.6$, (iv) $\sigma = 0.9$ and $\tau = 10^{-4}$

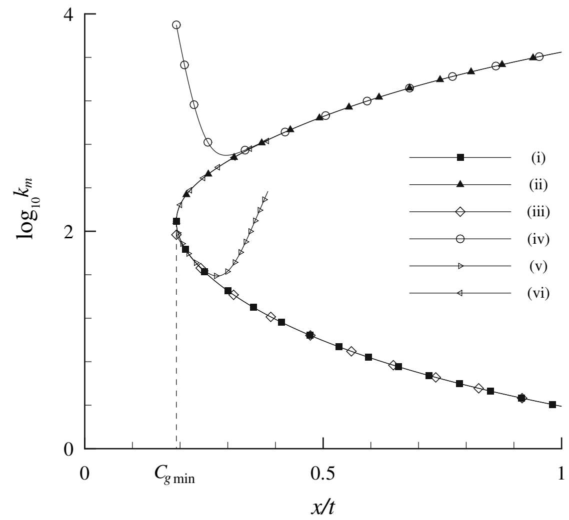

Fig. 2 Wavenumber curves $\log_{10} k_m$ with $\sigma = 0$. (i) exact solution for k_1, (ii) exact solution for k_2, (iii) asymptotic solution for k_1 with $|x|/t \gg C_{g\min}$, (iv) asymptotic solution for k_2 with $|x|/t \gg C_{g\min}$, (v) asymptotic solution for k_1 with $|x|/t \approx C_{g\min}$, and (vi) asymptotic solution for k_2 with $|x|/t \approx C_{g\min}$

where

$$X = 4x^2/(\lambda\tau t^2), \tag{40}$$

$$Q_1 = -6912G^2 + 288GX^2 + X^4, \tag{41}$$

$$Q_2 = -576G^3 + 36(GX)^2 + \sqrt{3G^3X^2Q_1}, \tag{42}$$

$$Q_3 = 48G - X^2, \tag{43}$$

$$Q_4 = 12(3Q_2)^{1/3} - 144G + 12GQ_3(9/Q_2)^{1/3} + X^2, \tag{44}$$

$$Q_5 = -6(3Q_2)^{1/3} - 144G + X^2 - 6GQ_3(9/Q_2)^{1/3} + (432GX + X^3)/Q_4^{1/2}. \tag{45}$$

The comparison between the exact solutions in Eq. 39 with $\sigma = 0$ and the asymptotic solutions of Chen and Duan [22, Eqs. 6 and 8] is illustrated in Fig. 2. It can be seen that Eqs. 6a,b and 8b,c of Chen and Duan [22] are asymptotic expressions valid for $|x|/t \gg C_{g\min}$ and $|x|/t \to C_{g\min}$, respectively. The exact solutions obtained here are uniformly valid for $|x|/t \geq C_{g\min}$ and show that k_1 and k_2 will tend to k_c at $|x|/t = C_{g\min}$, as is expected.

When $|x|/t > C_{g\min}$, according to the standard stationary-phase approximation [29, §3.4], the expansion for the phase function near k_m is taken as

$$\Theta_{lj}(k) \approx \Theta_{lj}(k_m) + \frac{1}{2}\frac{\partial^2\Theta_{lj}(k_m)}{\partial k^2}(k - k_m)^2. \tag{46}$$

A straightforward application of the stationary-phase approximation yields the formal expression for the viscous wave profile for large t with x/t held fixed,

$$\eta_1 \sim -\frac{M\sigma\lambda}{2}\sum_{m=1}^{2}\left(\frac{2}{\pi|\omega_m''|t}\right)^{1/2} d_m^T d_m^S \cos\psi_m, \tag{47}$$

$$\eta_2 \sim \frac{F\lambda}{2\rho_2}\sum_{m=1}^{2}\left(\frac{2}{\pi|\omega_m''|t}\right)^{1/2}\frac{d_m^T k_m}{\omega_m^2}\left[d_m^S\left(\omega_m\sin\psi_m+2\nu k_m^2\cos\psi_m\right)-2\nu k_m^2 d_m^B\cos\left(\psi_m-z_0\sqrt{\omega_m/2\nu}\right)\right], \quad (48)$$

where

$$d_m^T=\exp(-2\nu k_m^2 t), \quad (49)$$

$$d_m^S=\exp(-k_m|z_0|), \quad (50)$$

$$d_m^B=\exp(z_0\sqrt{\omega_m/2\nu}), \quad (51)$$

$$\psi_m=k_m|x|-\omega_m t+(-1)^{m+1}\pi/4, \quad (52)$$

$$\omega_m=(\gamma g k_m+\lambda\tau k_m^3)^{1/2}, \quad (53)$$

$$\omega_m''=\frac{\partial^2\omega(k_m)}{\partial k^2}=\frac{1}{4}(-G^2+6Gk_m^2+3k_m^4)\left[\frac{\lambda\tau}{(Gk_m+k_m^3)^3}\right]^{1/2}, \quad (54)$$

in which d_m^T is the temporal decay factor, and d_m^S and d_m^B are the submergence decay factors.

As $|x|/t$ decreases to approach $C_{g\min}$, k_1 and k_2 will go together toward the same limit k_c while ω_m'' tends to zero. Accordingly, Eqs. 47 and 48 predict that wave amplitudes will increase without bounds. It is noted that

$$\omega_m'''=\frac{\partial^3\omega(k_m)}{\partial k^3}=\frac{3}{8}(G^3+5G^2k_m^2-5Gk_m^4-k_m^6)\left[\frac{\lambda\tau}{(Gk_m+k_m^3)^5}\right]^{1/2}\neq 0. \quad (55)$$

To have a better approximation for Eq. 32 when k_m is close to k_c, we may expand the phase function near k_c as follows

$$\Theta_{lj}(k)\approx\Theta_{lj}(k_c)+\frac{\partial\Theta_{lj}(k_c)}{\partial k}(k-k_c)+\frac{1}{6}\frac{\partial^3\Theta_{lj}(k_c)}{\partial k^3}(k-k_c)^3. \quad (56)$$

Thus, according to Scorer [30], Eq. 32 can be approximated by

$$\eta_1\sim-\frac{M\sigma\lambda}{2}\left(\frac{2}{\omega_c'''t}\right)^{1/3}\mathrm{Ai}(Z_c^O)d_c^T d_c^S\cos\psi_c^O, \quad (57)$$

$$\eta_2\sim\frac{F\lambda}{2\rho_2}\left(\frac{2}{\omega_c'''t}\right)^{1/3}\mathrm{Ai}(Z_c^O)\frac{d_c^T k_c}{\omega_c^2}\left[d_c^S\left(\omega_c\sin\psi_c^O+2\nu k_c^2\cos\psi_c^O\right)-2\sqrt{\nu}k_c^2 d_c^B\cos\left(\psi_c^O-z_0\sqrt{\omega_c/2\nu}\right)\right], \quad (58)$$

where

$$\psi_c^O=k_c|x|-\omega_c t, \quad (59)$$

$$Z_c^O=(\omega_c' t-|x|)\left(\frac{2}{\omega_c'''t}\right)^{1/3}, \quad (60)$$

and $\mathrm{Ai}(z)$ is the Airy function. ω_c''' can be obtained from Eq. 55 by replacing k_m with k_c.

4 Three-dimensional interfacial waves due to an instantaneous singularity

For three-dimensional cases, the fundamental solution of the Laplace equation can be expressed as

$$\Phi_{11}^S=-\frac{M\delta(t)}{4\pi}\frac{1}{r}=-\frac{M}{16\pi^3\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\alpha\mathrm{d}\beta\frac{1}{K}\exp(-K|z-z_0|+f), \quad (61)$$

where $K = \sqrt{\alpha^2 + \beta^2}$, and $f = \mathrm{i}\alpha x + \mathrm{i}\beta y + st$. The fundamental solution of the unsteady Stokes equations can be written as

$$\mathbf{u}_{22}^S(\mathbf{x};\mathbf{x_0}) = \frac{1}{16\pi^3\mathrm{i}}\mathbf{F}\cdot(\nabla\nabla - \mathbf{I}\nabla^2)\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\alpha\mathrm{d}\beta\frac{1}{s}\exp(f) \times\left[\frac{1}{K}\exp(-K|z-z_0|) - \frac{1}{B}\exp(-B|z-z_0|)\right], \tag{62}$$

$$P_{22}^S(\mathbf{x};\mathbf{x_0}) = -\frac{1}{16\pi^3\mathrm{i}}(\mathbf{F}\cdot\nabla)\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\alpha\mathrm{d}\beta\frac{1}{K}\exp(-K|z-z_0| + f), \tag{63}$$

where $B = \sqrt{s/\nu + K^2}$. By taking a Laplace–Fourier transform similar to that in Eq. 23, we may get the following solutions for the three-dimensional problem:

$$\eta_n = \frac{1}{8\pi^3\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\alpha\mathrm{d}\beta\frac{\widehat{A}_n}{s\widehat{D}}\exp(f), \tag{64}$$

where $\widehat{A}_n$ and $\widehat{D}$ can be obtained from Eqs. 25–27 by replacing k and b with K and B, respectively. Furthermore, with a change of variables

$\{x, y\} = R\{\cos\theta, \sin\theta\},\ \{\alpha,\beta\} = K\{\cos\phi, \sin\phi\},$

Equation 64 can be re-written as

$$\eta_n = \frac{1}{4\pi^2\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_0^{+\infty}\mathrm{d}K\frac{KJ_0(KR)\widehat{A}_n}{s\widehat{D}}\exp(st), \tag{65}$$

where $J_0(KR)$ is the zeroth-order Bessel function of the first kind. Next, we may replace $J_0(KR)$ by its asymptotic formula for large KR [31, p. 364],

$$J_0(KR) \sim \left(\frac{2}{\pi KR}\right)^{1/2}\cos\left(KR - \frac{\pi}{4}\right). \tag{66}$$

Thus, we have an approximation for Eq. 65 as follows

$$\eta_n \sim \frac{1}{4\pi^2\mathrm{i}}\sum_{l=1}^{2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}s\int_0^{+\infty}\mathrm{d}k\left(\frac{k}{2\pi R}\right)^{1/2}\frac{A_n}{sD}\exp\left[(-1)^{l+1}\left(kR - \frac{\pi}{4}\right) + st\right]. \tag{67}$$

It is evident that Eq. 67 is the three-dimensional counterpart of Eq. 24. The oscillatory factors in the two- and three-dimensional problems are essentially the same, although the wave-amplitude factors are different. The asymptotic analysis for Eq. 67 is hence similar to that in Sect. 3, and the details can be omitted here.

When $R/t > C_{g\min}$, Eq. 67 can be approximated by

$$\eta_1 \sim -\frac{M\sigma\lambda}{2\pi}\sum_{m=1}^{2}\left(\frac{k_m}{R|\omega_m''|t}\right)^{1/2}d_m^T d_m^S\cos\varphi_m, \tag{68}$$

$$\eta_2 \sim \frac{F\lambda}{2\pi\rho_2}\sum_{m=1}^{2}\left(\frac{k_m}{R|\omega_m''|t}\right)^{1/2}\frac{d_m^T k_m}{\omega_m^2}[d_m^S(\omega_m\sin\varphi_m + 2\nu k_m^2\cos\varphi_m) - 2\nu k_m^2 d_m^B\cos(\varphi_m - z_0\sqrt{\omega_m/2\nu})], \tag{69}$$

where

$$\varphi_m = k_m R - \omega_m t + [(-1)^{m+1} - 1]\pi/4. \tag{70}$$

When $R/t \approx C_{g\min}$, Eq. 67 can be approximated by

$$\eta_1 \sim -\frac{M\sigma\lambda}{2}\left(\frac{k_c}{2\pi R}\right)^{1/2}\left(\frac{2}{\omega_c'''t}\right)^{1/3}\mathrm{Ai}(Z_c^O)d_c^T d_c^S\cos\varphi_c^O, \tag{71}$$

$$\eta_2 \sim \frac{F\lambda}{2\rho_2}\left(\frac{k_c}{2\pi R}\right)^{1/2}\left(\frac{2}{\omega_c'''t}\right)^{1/3}\mathrm{Ai}(Z_c^O)\frac{d_c^T k_c}{\omega_c^2}[d_c^S(\omega_c \sin\varphi_c^O + 2\nu k_c^2\cos\varphi_c^O)$$
$$-2\sqrt{\nu}k_c^2 d_c^B \cos(\varphi_c^O - z_0\sqrt{\omega_c/2\nu})], \quad (72)$$

where

$$\varphi_c^O = k_c R - \omega_c t - \pi/4. \quad (73)$$

5 Discussion and conclusions

We may check the present results by taking limits to recover the previous works. In the limit with $\sigma = \tau = 0$, Eq. 69 simply reduces to what Lu and Chwang [8] obtained for the viscous free-surface gravity waves due to a Stokeslet. Also, in the limit with $\sigma = \tau = \nu = h_0 = 0$, Eq. 48 simply reduces to what Stoker [26, p. 163] obtained by the classical potential theory for the inviscid free-surface waves generated by a line impulse at $x = 0$ and $t = 0$, while Eq. 69 agrees with that obtained by Stoker [26, p. 166] for inviscid concentric waves due to a concentrated point impulse at the origin and $t = 0$. The characteristics of the Cauchy–Poisson wave motion in an inviscid fluid have been discussed by Stoker [26] and Mei [27]. The classical Cauchy–Poisson problems are now extended in this study, incorporating the effects due to interfacial surface tension, an upper semi-infinite fluid, the submergence of a singularity, and the fluid viscosity.

Taking surface tension into consideration ($\tau > 0$), we have two real positive solutions for $|x|/t = C_g$ when $|x|/t > C_{g\min}$, one real positive solution when $|x|/t = C_{g\min}$ and no real solution when $|x|/t < C_{g\min}$. The two real solutions for the wavenumbers are associated with two wave systems, namely the long gravity-dominant ($m = 1$) and short capillary-dominant ($m = 2$) waves, which are predicted by Eqs. 47 and 48. The two waves are merged into one at $|x|/t = C_{g\min}$, which represents the wave front of the two wave systems. In the region with $|x|/t < C_{g\min}$, no wave motion will be observed since the minimum group velocity for the wave system occurs at $|x|/t = C_{g\min}$. However, the wave-like disturbances, described by Eqs. 57 and 58, should be produced so that a panoramic motion can be observed. Figures 3–6 show the wave profiles at different instants. Curves (i), (ii), (iii) and (iv) in Figs. 3–6 represent, respectively, the capillary–gravity waves predicted by Eq. 48 for $|x|/t > C_{g\min}$, the capillary–gravity wave elevation predicted by Eq. 58 for $|x|/t = C_{g\min}$, the capillary–gravity waves predicted by Eq. 58 for $|x|/t \approx C_{g\min}$ and $|x|/t < C_{g\min}$, and the pure gravity waves predicted by Lu and Chwang [8] for $|x|/t > 0$. Clearly, the amplitudes of the capillary waves are small in comparison with those of the gravity waves. The short capillary-dominant waves ride on the long gravity-dominant ones. As τ decreases, the capillary-dominant waves will be heavily damped. For small τ, the capillary-dominant waves are not significant and the generated waves are dominated by the gravity ones. When $\tau = 0$, $\omega(k)$ simply reduces to $\omega_0(k) = \sqrt{\gamma g k}$. $\omega_0''(k) < 0$ holds for $k > 0$. There is only one solution for $|x|/t = C_g$ with $|x|/t > 0$, $k_0 = gt^2/4x^2$, which corresponds to the pure gravity wave system.

It is known that pure gravity waves induced by a surface-piercing singularity in an inviscid fluid will oscillate rapidly with increasing amplitudes and decreasing wavelengths when the field point approaches the source point, as remarked by Chen and Duan [22]. This singular behavior can be suppressed when the viscosity of fluid [8] or the surface tension [22] is taken into account in the mathematical formulation. Analytical solutions for viscous gravity waves provided by Lu and Chwang [8] show that there exist temporal decay factors associated with the viscosity. Numerical analysis for inviscid capillary–gravity waves by Chen and Duan [22] shows the role of surface tension on the removal of the above-mentioned singular behavior. For inviscid fluids, the singular behavior can also be suppressed with the submergence decay factors, when the singularity is put beneath the free surface of a single fluid or the interface between two distinct fluids. The temporal decay factors and the submergence decay factors for pure gravity waves will tend to zero since the wave-numbers tend to infinity as the field point approaches the source point. Therefore, we may find, on the basis of the analytical solutions obtained here, a calm region on the free-surface or the interface that is near the submerged source point. This calm region will be enlarged due to the presence of surface tension, as shown in Figs. 3–6. The near-field capillary–gravity waves, as in

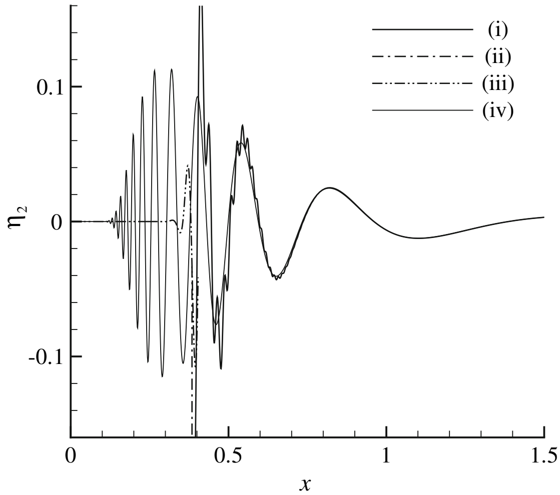

Fig. 3 Wave profiles (η_2) with $\sigma = 0$, $F/\rho_2 = 0.01$, $z_0 = -0.01$, $\nu = 10^{-6}$, $\tau = 10^{-4}$, and $t = 2$

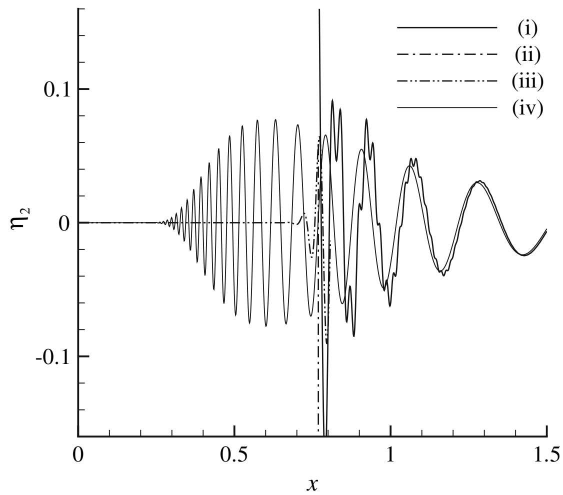

Fig. 4 Wave profiles (η_2) with $\sigma = 0$, $F/\rho_2 = 0.01$, $z_0 = -0.01$, $\nu = 10^{-6}$, $\tau = 10^{-4}$, and $t = 4$

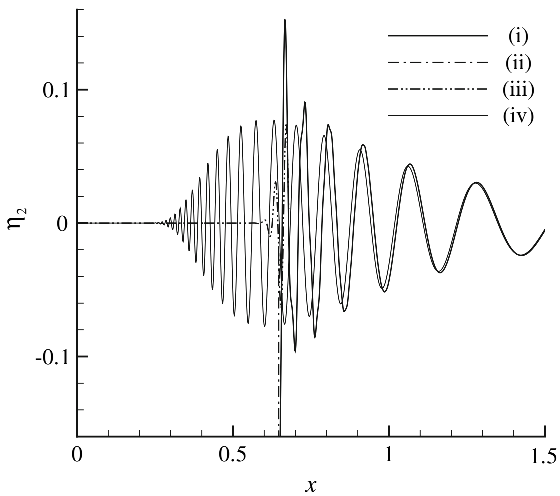

Fig. 5 Wave profiles (η_2) with $\sigma = 0$, $F/\rho_2 = 0.01$, $z_0 = -0.01$, $\nu = 10^{-6}$, $\tau = 5 \times 10^{-5}$, and $t = 4$

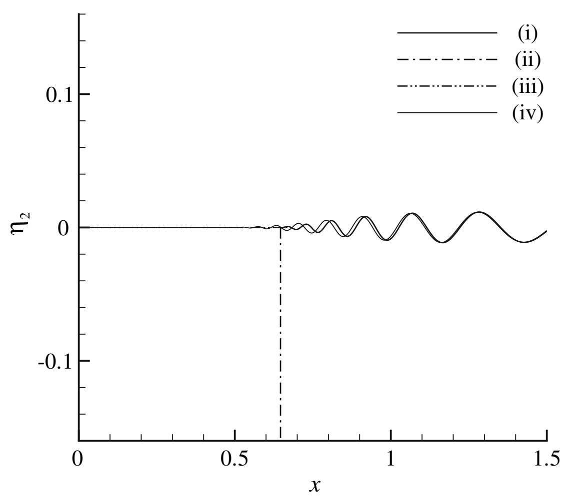

Fig. 6 Wave profiles (η_2) with $\sigma = 0$, $F/\rho_2 = 0.01$, $z_0 = -0.05$, $\nu = 10^{-6}$, $\tau = 5 \times 10^{-5}$, and $t = 4$

Eqs. 57 and 58, are heavily damped due to the presence of the Airy function $\mathrm{Ai}(Z_c^O)$. As time increases, the waves propagate outwards and the calm region is extended. As τ increases, the calm region is also extended.

The waves generated are damped due to the presence of the exponential decay factors, d_m^T, d_m^S and d_m^B, which are associated with the wavenumbers, the viscosity of the lower fluid, and the submergence depth of the singularity. Figure 7 shows the wave evolution as a function of time for a fixed position for the case shown in Fig. 5. In the initial stage, the resultant waves are essentially the pure gravity waves. It is apparent from Fig. 1 that k_2 is very large when x/t is large. Thus the capillary-waves associated with k_2 are heavily attenuated. As time increases, the amplitude of the capillary waves will first increase toward its maximum at $t = x/C_{g\min}$, and then diminish to become zero when $t > x/C_{g\min}$. The theoretical results obtained by the present asymptotic analysis agree well with the direct numerical results obtained by Chen and Duan [22] for $\sigma = 0$. Figure 8 shows the effect of density ratio (σ) on the

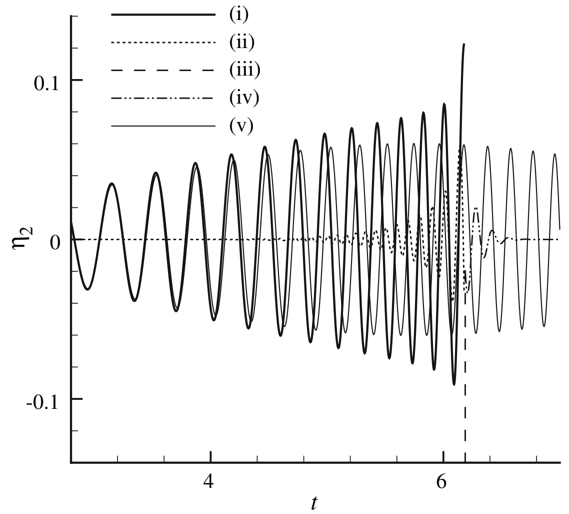

Fig. 7 Wave profiles (η_2) with $\sigma = 0$, $F/\rho_2 = 0.01$, $z_0 = -0.01$, $\nu = 10^{-6}$, $\tau = 5 \times 10^{-5}$, and $x = 1$. (i) gravity-dominant waves predicted by Eq. 48 for $t < x/C_{g\min}$, (ii) capillary-dominant waves predicted by Eq. 48 for $t < x/C_{g\min}$, (iii) resultant waves predicted by Eq. 58 at $t = x/C_{g\min}$, (iv) resultant waves predicted by Eq. 58 at $t > x/C_{g\min}$, and (v) pure gravity waves

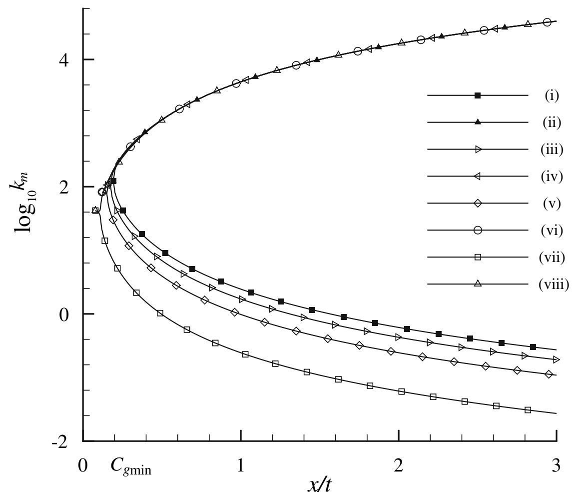

Fig. 8 Wavenumber curves $\log_{10} k_m$ with (i)–(ii) $\sigma = 0$, (iii)–(iv) $\sigma = 0.3$, (v)–(vi) $\sigma = 0.6$, and (vii)–(viii) $\sigma = 0.9$. The upper lines are for capillary-dominant ($m = 2$) waves with $\tau = 5 \times 10^{-4}$ while the lower ones for gravity-dominant ($m = 1$) waves

wavelengths. The wavelengths on the interface are elongated while the amplitudes are attenuated, in comparison with those of free-surface waves in a single fluid.

In this paper, a system consisting of an upper inviscid and a lower viscous fluid has been considered. The work can be readily extended to other similar two-layer systems, such as one with both fluids being inviscid or viscous, or one with a viscous fluid overlying an inviscid fluid. The cases with two-layer fluids of finite depth are also worth pursuing.

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